Math for Management, Winter 2023

List 5

Derivative calculations

For a function $f(x)$ and a number a, the **derivative of f at a**, written $f'(a)$, is the slope of the tangent line to $y = f(x)$ at the point $(a, f(a))$ and is calculated as

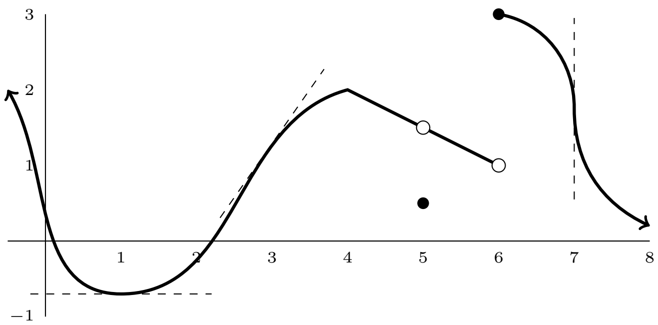
$$f'(a) \quad = \quad \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x} \quad = \quad \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

The function $f(x)$ is **differentiable at a** if $f'(a)$ exists and is finite.

104. Calculate $f'(5)$ for the function $f(x) = x^3$. Hint: See Task 90(b). $\boxed{75}$

105. Calculate $f'(1)$ for the function $f(x) = \sqrt{x}$. Hint: See Task 50(b). $\boxed{\frac{1}{2}}$

106. The graph of a function is shown below. Near $x = 1$, $x = 3$, and $x = 7$, part of the tangent lines to the graph at those points is shown as a dashed line segment.



(a) List all points where the function is not continuous. $\boxed{x = 5, x = 6}$

(b) List all points where the function is not differentiable (that is, where the derivative does not exist). $\boxed{x = 4, x = 5, x = 6, x = 7}$

107. List all points where $f(x) = \dfrac{|x| - 4}{|x - 4|}$ is not differentiable. $\boxed{x = 0 \text{ and } x = 4}$ The function has a "corner" at $x = 0$, so it is continuous there but not differentiable. It has a "jump" at $x = 4$, so it is not continuous and also not differentiable there.

The Constant Multiple Rule: If c is a constant then

$$(c\,f)' = c\,f' \qquad \big(c\,f(x)\big)' = c\,f'(x) \qquad \tfrac{\mathrm{d}}{\mathrm{d}x}\big[c\,f\big] = c\,\tfrac{\mathrm{d}f}{\mathrm{d}x} \qquad D\big[c\,f\big] = c\,D[f]$$

(these are four ways of writing exact the same fact).

The Sum Rule: $\frac{\mathrm{d}}{\mathrm{d}x}\big[f + g\big] = \frac{\mathrm{d}}{\mathrm{d}x}[f] + \frac{\mathrm{d}}{\mathrm{d}x}[g]$.

The Power Rule: If p is a constant then $\frac{\mathrm{d}}{\mathrm{d}x}\big[x^p\big] = p\,x^{p-1}$.

108. All parts of this task have exactly the same answer! Answer: $\boxed{14x^6}$

(a) Find $f'(x)$ for the function $f(x) = 2x^7$.

(b) Give f' if $f = 2x^7$.

(c) Find y' for $y = 2x^7$.

(d) Compute $\frac{\mathrm{d}f}{\mathrm{d}x}$ for the function $f(x) = 2x^7$.

(e) Compute $\frac{\mathrm{d}y}{\mathrm{d}x}$ for $y = 2x^7$.

(f) Give the derivative of $2x^7$ with respect to x.

(g) Find the derivative of $2x^7$.

(h) Calculate $\frac{\mathrm{d}}{\mathrm{d}x}2x^7$. (i) Calculate $\left(2x^7\right)'$. (j) Calculate $D\left[2x^7\right]$.

(k) Differentiate $2x^7$ with respect to x.

(ℓ) Differentiate $2x^7$.

109. Differentiate $x^5 + \frac{2}{9}x^3 + \sqrt{3x} + \dfrac{x^{10}}{\sqrt{x}}$. $\boxed{5x^4 + \dfrac{2}{3}x^2 + \dfrac{\sqrt{3}}{2\sqrt{x}} + \dfrac{19}{2}x^{17/2}}$

110. Differentiate $\left(x + \sqrt{x}\right)^2$. $\boxed{2x + 3\sqrt{x} + 1}$ or $2\left(x + \sqrt{x}\right)\left(1 + \frac{1}{2\sqrt{x}}\right)$

111. For each of the functions below, can the Power Rule and/or Constant Multiple Rule (along with maybe some algebra) be used to find the derivative? If so, give the derivative.

(a) $2x^6$ $\boxed{\text{Yes: } 12x^5}$

(b) $2\sqrt{x}$ $\boxed{\text{Yes: } x^{-1/2}}$ or $\dfrac{1}{\sqrt{x}}$

(c) $\sqrt{5x}$ $\boxed{\text{Yes: } \frac{\sqrt{5}}{2}x^{-1/2}}$

(d) x^π $\boxed{\text{Yes: } \pi x^{\pi-1}}$

(e) $x^{\sin x}$ $\boxed{\text{No}}$

(f) $(\sin x)^x$ $\boxed{\text{No}}$

(g) e^x $\boxed{\text{No!}}$

(h) $\cos(5x)$ $\boxed{\text{No}}$

(i) $\sin(5\cos(x))$ $\boxed{\text{No}}$

(j) $e^{5\ln(x)}$ $\boxed{\text{Yes: } 5x^4}$

(k) $\dfrac{3}{x^6}$ $\boxed{\text{Yes: } -18x^{-7}}$

(ℓ) x^x $\boxed{\text{No!}}$

(m) $\ln(2 + x)$ $\boxed{\text{No}}$

(n) $\ln(2x)$ $\boxed{\text{No}}$

(o) $\ln(2^x)$ $\boxed{\text{Yes: } \ln(2)}$

(p) $\ln(x^2)$ $\boxed{\text{No}}$

112. Is it possible to find the derivative of the following functions using the Power Rule, Constant Multiple Rule, and Sum Rule?

(a) $x + \ln(5e^x)$ This function equals $2x + \ln(5)$, so $\boxed{\text{Yes: } 2}$.

(b) $\dfrac{2x}{x+6}$ No

(c) $\dfrac{x+6}{2x}$ Yes: $-3x^{-2}$ or $\frac{-3}{x^2}$

(d) $\dfrac{x+\frac{1}{x}}{\sqrt{x}}$ Yes: $\frac{1}{2}x^{-1/2} - \frac{3}{2}x^{-5/2}$

113. Give the derivative of each of the following functions.

(a) x^{7215} $7215x^{7214}$

(b) $5x^{100} + 9x$ $500x^{99} + 9$

(c) $2x^3 - 6x^2 + 10x + 1$ $6x^2 - 12x + 10$

(d) $3\sqrt{x}$ $\frac{3}{2}x^{-1/2}$ or $\dfrac{3}{2\sqrt{x}}$

(e) $\sqrt[3]{x}$ $\frac{1}{3}x^{-2/3}$ or $\dfrac{1}{3\sqrt[3]{x^2}}$

(f) $\sqrt{x}^3$ $\frac{2}{3}x^{-1/3}$ or $\dfrac{2}{3\sqrt[3]{x}}$

(g) 31 0

(h) $x + \frac{1}{x}$ $1 - x^{-2}$ or $1 - \dfrac{1}{x^2}$

(i) $\sqrt{x} + \frac{1}{\sqrt{x}}$ $\frac{1}{2}x^{-1/2} + \frac{-1}{2}x^{-3/2}$ or $\dfrac{1}{2\sqrt{x}} - \dfrac{1}{2\sqrt{x^3}}$

(j) $(3x+7)^2$ $18x + 42$ or $6(3x+7)$

114. Is $x^3 - x^{1/3}$ continuous everywhere? Yes Is it differentiable everywhere? No because $\frac{dy}{dx}$ does not exist at $x = 0$.

115. If $f(x) = 8x^4 - x^2$, for what values of x does $f(x) = 0$? $\dfrac{-1}{2\sqrt{2}}, 0, \dfrac{1}{2\sqrt{2}}$

For what values of x does $f'(x) = 0$? $\dfrac{-1}{4}, 0, \dfrac{1}{4}$

116. For the function $f(x) = x^3$ and $g(x) = 2x^2$, ...

(a) Calculate the derivative of f. $3x^2$

(b) Calculate the derivative of g. $4x$

(c) Calculate the derivative of

$$f(x) + g(x) = x^3 + 2x^2.$$

$3x^2 + 4x$

(d) Calculate the derivative of

$$f(x) \cdot g(x) = 2x^5.$$

$\boxed{10x^4}$

(e) Does $(f+g)' = f' + g'$? In other words, is your answer to (c) the same as adding your answers to (a) and (b)? $\boxed{\textbf{Yes}}$

(f) Does the derivative of a sum equal the sum of the derivatives? $\boxed{\textbf{Yes}}$

(g) Does $(f \cdot g)' = f' \cdot g'$? In other words, is your answer to (d) the same as multiplying your answers to (a) and (b)? $\boxed{\textbf{No!}}$

(h) Does $\frac{\mathrm{d}}{\mathrm{d}x}\left[f \cdot g\right] = \frac{\mathrm{d}f}{\mathrm{d}x} \cdot \frac{\mathrm{d}g}{\mathrm{d}x}$? This is exactly the same question as (g). $\boxed{\textbf{No!}}$

(i) Does the derivative of a product equal the product of the derivatives? $\boxed{\textbf{No!}}$

117. Which limit expression below gives the derivative of x^3 at the point $x = 2$? $\boxed{\text{(C)}}$

(A) $\displaystyle\lim_{x\to 2} \frac{x^3 - 8}{x}$ (C) $\displaystyle\lim_{h\to 0} \frac{(2+h)^3 - 8}{h}$

(B) $\displaystyle\lim_{h\to 0} \frac{h^3 - 8}{h}$ (D) $\displaystyle\lim_{h\to 0} \frac{(2+h)^3 - h^3}{h}$

118. (a) Find $\left(x^{10} + 100x + 1000\right)' = \boxed{10x^9 + 100}$

(b) Find $D\left[9x + \sqrt{9x}\right] = D\left[9x + 3x^{1/2}\right] = \boxed{9 + \frac{3}{2}x^{1/2}}$

(c) Find $\dfrac{\mathrm{d}}{\mathrm{d}x}\left[(2x+3)^2\right] = \dfrac{\mathrm{d}}{\mathrm{d}x}\left[4x^2 + 12x + 9\right] = \boxed{8x + 12}$

(d) Find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ for $y = \dfrac{x+12}{2x}$. $\dfrac{\mathrm{d}}{\mathrm{d}x}\left[\dfrac{x+12}{2x}\right] = \dfrac{\mathrm{d}}{\mathrm{d}x}\left[\dfrac{1}{2} + \dfrac{6}{x}\right] = 0 + \boxed{\dfrac{-6}{x^2}}$

119. For each function below, state whether its derivative can be found using *only* algebra, the Power Rule, the Constant Multiple Rule, and the Sum Rule. If so, give its derivative.

(a) $4x^2 - 27x$ has derivative $\boxed{8x - 27}$.

(b) $4x^2 - 27$ has derivative $\boxed{8x}$. Note that $27 = 27x^0$, so we can use the Power Rule to get $27 \cdot 0x^{-1} = 0$ as its derivative.

(c) $\sqrt{16x}$ is equal to $4x^{1/2}$ and therefore has derivative $\boxed{2x^{-1/2}}$, or $\boxed{\frac{2}{\sqrt{x}}}$.

(d) $(x + \sqrt{7})^2$ is equal to $x^2 + 2\sqrt{7}x + 7$ and therefore has derivative $\boxed{2x + 2\sqrt{7}}$.

(e) 2^{x+7} is equal to $128 \cdot 2^x$, but differentiating this requires another rule.

(f) $\dfrac{5}{x}$ is equal to $5x^{-1}$ and therefore has derivative $\boxed{-5x^{-2}}$, or $\boxed{\frac{-5}{x^2}}$.

(g) $\dfrac{3x}{6x+15}$ requires the Quotient Rule (or the Chain and Product Rules together).

(h) $\dfrac{6x+15}{3x}$ is equal to $2 + \frac{5}{x}$ and therefore has derivative $\boxed{\frac{-5}{x^2}}$.

Individual functions: $\frac{\mathrm{d}}{\mathrm{d}x}\big[x^p\big] = px^{p-1}$, $\frac{\mathrm{d}}{\mathrm{d}x}\big[e^x\big] = e^x$, $\frac{\mathrm{d}}{\mathrm{d}x}\big[\ln(x)\big] = \frac{1}{x}$, $\frac{\mathrm{d}}{\mathrm{d}x}\big[\sin(x)\big] = \cos(x)$, $\frac{\mathrm{d}}{\mathrm{d}x}\big[\cos(x)\big] = -\sin(x)$.
Sum Rule: $(f+g)' = f' + g'$ **Product Rule:** $(f \cdot g)' = fg' + f'g$
Chain Rule: $(f(g))' = f'(g) \cdot g'$ **Quotient Rule:** $(f/g)' = \dfrac{gf' - fg'}{g^2}$

120. Give the derivative of $5\sin(x) + \frac{2}{3}\cos(x) - x^3 + 9$. $\boxed{5\cos(x) - \frac{2}{3}\cos(x) - 3x^2}$

121. Using the Product Rule, give the derivative of $5^x \cdot \sin(x)$. $\boxed{5^x \cos(x) + 5^x \ln(x)\sin(x)}$

122. Use the Product Rule (twice) to find the derivative of $x^6 \cdot \cos(x) \cdot 2^x$.
$x^6\cos(x)2^x\ln(2) - x^6\sin(x)2^x + 6x^5\cos(x)2^x$ can be simplified to
$\boxed{x^5 2^x\big(x\cos(x)\ln(2) - x\sin(x) + 6\cos(x)\big)}$

123. Give the derivative of every function in Task 119.

124. True or false?
   (a) $(f+g)' = f' + g'$ $\boxed{\text{true}}$
   (b) $(f \cdot g)' = f' \cdot g'$ $\boxed{\text{false}}$ A correct right-hand side could be parts (c) or (d).
   (c) $(f \cdot g)' = f'g + fg'$ $\boxed{\text{true}}$
   (d) $\frac{\mathrm{d}}{\mathrm{d}x}(fg) = f\frac{\mathrm{d}g}{\mathrm{d}x} + g\frac{\mathrm{d}f}{\mathrm{d}x}$ $\boxed{\text{true}}$
   (e) $(f \cdot g)' = g'f' + gf'$ $\boxed{\text{false}}$ A correct right-hand side could be $g'f + gf'$ without the extra $'$ in the first term $g'f'$.
   (f) $(f/g)' = gf' - fg'$ $\boxed{\text{false}}$ A correct right-hand side could be $(gf' - fg')/g^2$.

125. Find the derivative of $\sin(5^{\cos(2x^3+8)})$.
$\boxed{\cos(5^{\cos(2x^3+8)}) \cdot 5^{\cos(2x^3+8)}\ln(5) \cdot \big(-\sin(2x^3+8)\big) \cdot 6x^2}$

126. (a) Use the Quotient Rule to differentiate $\dfrac{\sin(x)}{x^4}$. $\boxed{\dfrac{x^4\cos(x) - \sin(x)(4x^3)}{x^8}}$
   (b) Use the Product Rule to differentiate $x^{-4}\sin(x)$. $\boxed{x^{-4}\cos(x) + (-4x^{-5})\sin(x)}$
   (c) Use algebra to compare your answers from parts (a) and (b). $\boxed{\text{they are equal}}$

127. Find the following derivatives (note (p)-(ż) require the Chain Rule).
   (a) $f'(x)$ for $f(x) = x^9\sin(x)$ $\boxed{x^9\cos(x) + 9x^8\sin(x)}$
   (ą) $\frac{\mathrm{d}}{\mathrm{d}x}\big(x^9\sin(x)\big)$ $\boxed{x^9\cos(x) + 9x^8\sin(x)}$ (same as (a))
   (b) $\frac{\mathrm{d}}{\mathrm{d}x}\big(10^x + \log_{10}(x)\big)$ $\boxed{10^x\ln(10) + \dfrac{1}{x\ln(10)}}$
   (c) $\frac{\mathrm{d}}{\mathrm{d}x}\big(10^x \cdot \log_{10}(x)\big)$ $\boxed{10^x\ln(x) + \dfrac{10^x}{x\ln(10)}}$

(ć) $\frac{d}{dx}\left(\sqrt{x}\sin(x)\right)$ $\boxed{\dfrac{\sin(x)}{2\sqrt{x}} + \sqrt{x}\cos(x)}$

(d) $\frac{d}{dx}\left(x^9\, e^x \sin(x)\right)$ $\boxed{e^x x^9 \sin(x) + e^x x^9 \cos(x) + 9e^x x^8 \sin(x)}$

(e) $\frac{d}{dx}(4x^3 + x\sin x)$ $\boxed{12x^2 + \sin(x) + x\cos(x)}$

(ę) $\frac{d}{dt}(4t^3 + t\sin t)$ $\boxed{12t^2 + \sin(t) + t\cos(x)}$

(f) $\frac{d}{dt}\ \sin(t)\cos(t)$ $\boxed{(\cos x)^2 - (\sin x)^2} = \cos(2x)$

(g) $\dfrac{d}{dx}\ \dfrac{\cos(x)}{5x^3 - 12}$ $\boxed{\dfrac{-(5x^3-12)\sin(x) - 15x^2\cos(x)}{(12-5x^3)^2}}$

(h) $\dfrac{d}{dx}\dfrac{5x^3 - 12}{\cos(x)}$ $\boxed{(5x^3 - 12)\sec(x)\tan(x) + 15x^2\sec(x)}$

(i) $\dfrac{d}{dt}\dfrac{t^7 + t^2}{e^t}$ $\boxed{-e^{-t}t(t^6 - 7t^5 + t - 2)}$

(j) $\frac{d}{dx}(5x-7)^2$ $\boxed{50x - 70}$

(k) $\frac{d}{dt}\ e^t\cos(t)$ $\boxed{e^t\cos(t) - e^t\sin(t)}$

(l) $\dfrac{d}{dt}\left(t\sin(t) + \dfrac{e^t}{t^2+1}\right)$ $\boxed{\dfrac{e^t(t-1)^2}{(t^2+1)^2} + \sin(t) + t\cos(t)}$

(ł) $\dfrac{d}{dx}\dfrac{\sin(t)}{t\, e^t}$ $\boxed{\dfrac{t\cos(t) - (t+1)\sin(t)}{t^2 e^t}}$

(m) $\frac{d}{dt}\ t^{5/2}\sin(t)$ $\boxed{\dfrac{5}{2}t^{3/2}\sin(t) + t^{5/2}\cos(t)}$

(n) $\frac{d}{dx}\ 2^{15}$ $\boxed{0}$

(ń) $\frac{d}{dx}\ x^{15}$ $\boxed{15x^{14}}$

(o) $\frac{d}{du}\ u^{15}$ $\boxed{15u^{14}}$

(ó) $\frac{d}{dx}\ u^{15}$ if u is a constant $\boxed{0}$

(p) $\frac{d}{dx}\ u^{15}$ if u is a fn. of x $\boxed{15u^{14}\frac{du}{dx}}$

(q) $\frac{d}{dx}(\cos(x))^{15}$ $\boxed{15(\cos x)^{14}(-\sin t)}$

(r) $\frac{d}{dx}\ln(\cos(x))$ $\boxed{-\tan(x)}$

(s) $\frac{d}{dx}\sqrt{\ln(\cos(x))}$ $\boxed{\dfrac{-\tan(x)}{2\sqrt{\ln(\cos(x))}}}$

(ś) $\dfrac{d}{dx}\ e^{\sqrt{\ln(\cos(x))}}$ $\boxed{\dfrac{-\tan(x)e^{\sqrt{\ln(\cos(x))}}}{2\sqrt{\ln(\cos(x))}}}$

(t) $\dfrac{\mathrm{d}}{\mathrm{d}x}\, e^{\sqrt{\ln(\cos(x^6))}}$ $\boxed{\dfrac{-3x^5\tan(x^6)e^{\sqrt{\ln(\cos(x^6))}}}{\sqrt{\ln(\cos(x^6))}}}$

(u) $\frac{\mathrm{d}}{\mathrm{d}t}\, 5\sin(2t+1)$ $\boxed{10\cos(2t+1)}$

(v) $\frac{\mathrm{d}}{\mathrm{d}t}\, A\sin(\omega t+\phi)$ if A, ω, t are constants $\boxed{A\omega\cos(\omega t+\phi)}$

(w) $\frac{\mathrm{d}}{\mathrm{d}x}(7x^2+\sin(x))^2$ $\boxed{2(7x^2+\sin(x))(14x-\cos(x))}$

(x) $\frac{\mathrm{d}}{\mathrm{d}x}(\log_3(x))^2$ $\boxed{(2\ln(x)))/(x\ln(3)^2)}$

(y) $\frac{\mathrm{d}}{\mathrm{d}t}\tan(t^3+8t^2+2t+18)$ $\boxed{(3t^2+16t+2)\big(\sec(t^3+8t^2+2t+18)\big)^2}$

(z) $\frac{\mathrm{d}}{\mathrm{d}x}\cos(x^3e^x)$ $\boxed{3x^2\cos(9x)-9x^3\sin(9x))}$

(ź) $\frac{\mathrm{d}}{\mathrm{d}x}\, x^3\cos(9x)$ $\boxed{3x^2\big(\cos(9x)-3x\sin(9x)\big)9}$

(ż) $\dfrac{\mathrm{d}}{\mathrm{d}x}\dfrac{x^3\cos(x)}{e^{\sin(x)}}$ $\boxed{x^2(-e^{-\sin(x)})\big(x\sin(x)+\cos(x)(x\cos(x)-3)\big)}$